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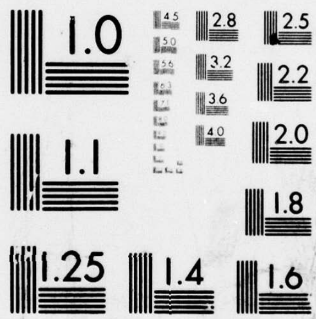

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STRUCTURAL INELASTICITY XXIV

Examples of Non-Uniqueness in Contained Plastic Deformation

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It is well known that in a well-defined boundary-value problem for an elastic/perfectly-plastic structure the displacements are unique if the structure is everywhere elastic, and they are not unique at the yield-point load. The present paper is concerned with the range of contained plastic deformation between these two extremes. Several examples are given in which more than one displacement field exists for loads less . . .

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NON-UNIQUENESS IN CONTAINED PLASTIC DEFORMATION *

by

Philip G. Hodge, Jr. **

and

David L. White ***

Abstract

It is well known that in a well-defined boundary-value problem for an elastic/perfectly-plastic structure the displacements are unique if the structure is everywhere elastic, and they are not unique at the yield-point load. The present paper is concerned with the range of contained plastic deformation between these two extremes. Several examples are given in which more than one displacement field exists for loads less than the yield-point load. The significance of this phenomenon is commented on from a physical, mathematical, and computational point of view.

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I. Introduction

The solution of boundary value problems in the mechanics of solids is facilitated by making assumptions about the material and its response to prescribed boundary conditions. These assumptions lead to a mathematical model which approximates the response of the physical system. The accuracy of the approximation varies with the system in question, but frequently much can be learned about a complex system by studying a simple model. However, a simple model may exhibit qualitative phenomena which are not present in the physical system.

The elastic-perfectly plastic model is one such simple model which is useful between the elastic limit and the yield point load. At any fixed point such a material responds elastically until the stresses satisfy the yield function. As long as the stresses continue to satisfy the yield function the material responds plastically with no strain hardening. This paper considers elastic-perfectly plastic structures subjected to a system of loads defined by a single monotonically increasing parameter λ. The structure behaves elastically up to some value of λ, λ_e. For larger values of λ a portion of the structure will be plastic and a portion will be elastic. The elastic portion will limit the displacements and the structure is said to be in the range of contained plastic deformation. At the yield point load, $\lambda = \lambda_y$, deformations can increase indefinitely (assuming geometry changes are ignored). The structure is in the stage of unrestricted plastic flow for the constant load λ_y.

The magnitude of the deformation at $\lambda = \lambda_y$ is not determined, hence the solution is non-unique. It will often be unique up to a single magnitude parameter, but many examples are known of systems with two or more degrees of kinematic freedom at the yield point load.

Considerable theoretical work has been done on the question of uniqueness in elastic-plastic problems. Hill[1] has shown that there is always a unique solution when the total strain is small, the work hardening is monotonic and not zero, and the yield function and plastic potential are identical. Hill[2] has also demonstrated the following necessary condition for the existence of two distinct solutions:

$$I \equiv \int_V (\dot{\sigma}' - \dot{\sigma}'')(\dot{\varepsilon}' - \dot{\varepsilon}'')\, dV = 0 \qquad (1)$$

For an elastic-perfectly plastic structure at the yield point load, I = 0. This accomodates the known lack of uniqueness. For contained plastic deformation, I may be positive or zero. If the assumption of two distinct solutions necessarily leads to I being positive, then the solution is unique. Otherwise the question is open.

The purpose of the present paper is to present several simple structures where I = 0 and more than one solution does in fact exist in the range of contained plastic deformation. The first example is a simple three bar truss which has frequently been used in the literature[3,4] to illustrate phenomena which also occur in more complex cases. Section 2 offers a solution to this truss which explicitly demonstrates the lack of uniqueness.

*Superscript numbers refer to the list of references on page 20.

Section 3 carefully examines the characteristics associated with this lack of uniqueness from physical, mathematical, and computational viewpoints. Even though the three bar truss is a simple example, it leads one to observe several general properties defining this type of non-uniqueness.

Section 4 tests these observations on further examples involving trusses, frames, and arches. In each of these examples the general properties proposed in section 3 are seen to hold. Finally, section 5 discusses these results and the directions which further investigation may take.

An appendix to the report gives some of the details of the examples considered in Sec. 4.

II. SIMPLE TRUSS

The three bar truss in Figure 1 offers insight into the non-uniqueness phenomenon and is particularly easy to deal with mathematically. The two-force members OA, OB, and OC join the point O to the fixed points A, B, and C. The structure is symmetric about OB. The load P increases monotonically from zero such that the system is quasi-static. The subscripts 1, 2, and 3 refer to the bars OA, OB, and OC respectively. Young's modulus E, and the area A are the same for each bar. The yield strength of bar OB is $3\sigma_y$. The yield strength of bars OA and OC is σ_y.

This problem is statically indeterminant to the first degree. There are three types of equations which define the problem: equilibrium, compatibility, and constitutive.

Horizontal and vertical equilibrium of the joint O give

$$F_1 = F_3 \tag{2a}$$

$$F_2 + (F_1 + F_3)/\sqrt{2} = P \tag{2b}$$

Compatibility is assured if the extensions of the bars are defined in terms of u and v, the horizontal and vertical (small) displacements of O:

$$\Delta_1 = (u + v)/\sqrt{2} \tag{3a}$$

$$\Delta_2 = v \tag{3b}$$

$$\Delta_3 = (-u + v)/\sqrt{2} \tag{3c}$$

The constitutive equations may be expressed in several ways.

For the present example where all bar forces are positive and non-decreasing they can be written in the simple form:

$$F_i \leq Y_i \tag{4a}$$

$$\text{If } F_i < Y_i \text{ then } \Delta_i = (L_i/AE)F_i \tag{4b}$$

$$\text{Else } F_i = Y_i \text{ and } \Delta_i \geq (L_i/AE)Y_i \tag{4c}$$

where $Y_1 = Y_3 = A\sigma_y$ and $Y_2 = 3A\sigma_y$ are the yield forces in the bars, and $L_1 = L_3 = \sqrt{2}\,H$ and $L_2 = H$ are the lengths.

For P suffieciently small, all bars are elastic and the branch (4b) applies to all three bars. Equations (3) are regarded as definitions of Δ_i, whence Equations (2) and (4b) form a system of five non-homogeneous linear equations in five unknowns: F_1, F_2, F_3, u, and v. The solution of this system is

$$F_1 = F_3 = P/(2 + \sqrt{2}) \qquad F_2 = 2P/(2 + \sqrt{2}) \tag{5}$$

$$u = 0 \qquad v = F_2H/AE$$

This solution provides the values for stress, strain and displacement at every point in the structure. The solution is unique in that it is the only set of forces and displacements which satisfies the governing equations. The solution is valid so long as the inequality in (4b) holds for each bar, i. e., until one or more of the bars yields. For the value

$$P_e = (2 + \sqrt{2})\sigma_y A \tag{6}$$

bars OA and OC yield and equation (4b) is no longer valid for bars 1 and 3.

As P is further increased, branch (4c) must be used for the two yielded bars. The governing equations are (2) and

$$F_1 = A\sigma_y \tag{7a}$$

$$v = \Delta_2 = (H/AE)F_2 \tag{7b}$$

$$F_3 = A\sigma_y \tag{7c}$$

The four equations (2), (7a), and (7c) contain only the unknowns F_1, F_2, and F_3. However, the system is consistent and the unique solution is

$$F_1 = F_3 = A\sigma_y \qquad F_2 = P - \sqrt{2}A\sigma_y \tag{8a}$$

Combination of this result with (7b) gives

$$v = (H/AE)\ (P - \sqrt{2}A\sigma_y) \tag{8b}$$

The inequality in (4b) must hold for bar 2, and the inequality in (4c) must hold for bars 1 and 3. Therefore

$$P < (3 + \sqrt{2})A\sigma_y \tag{8c}$$

$$-\ (H/AE)\ (P - P_e) \leq u \leq (H/AE)\ (P - P_e) \tag{8d}$$

where P_e is given by (6).

Thus in the range of contained plastic deformation, i. e.

$$(2 + \sqrt{2})A\sigma_y \leq P < (3 + \sqrt{2})A\sigma_y \tag{9}$$

(8) provides a complete set of solutions of the system of governing equations and inequalities. Although F_1, F_2, F_3. and v are single valued, u is not uniquely specified. The

range of u is largest at the yield point load. In the limit as P approaches P_y, with σ_y = 30 ksi and E = 30,000 ksi, v and u are given by

$$v/H = .003 \qquad -.001 \leq u/H \leq .001$$

Thus, the range of values for u is the same order of magnitude as the vertical deflection.

At the yield point load

$$P = P_y = (3 + \sqrt{2})A\sigma_y \tag{10}$$

the branch (4c) holds for all three bars and the forces and displacements satisfy:

$$F_1 = F_3 = A\sigma_y \qquad F_2 = 3A\sigma_y \tag{11a}$$

$$v \geq 3H\sigma_y/E \qquad -(v - 2H\sigma_y E) \leq u \leq (v - 2H\sigma_y/E) \tag{11b}$$

III. ANALYSIS

In order to see the salient features of the previous example with a view to later generalization, it is desirable to write Equations (2) and (3) in rate form, and to rewrite the constitutive equations as

$$F_i^2 \leq Y_i^2 \tag{12a}$$

$$\text{If} \quad F_i^2 < Y_i^2 \quad \text{or} \quad F_i\dot{F}_i < 0 \quad \text{then} \quad \dot{\Delta}_i = (L_i/AE)\dot{F}_i \tag{12b}$$

$$\text{Else} \quad \dot{F}_i = 0 \quad \text{and} \quad F_i\dot{\Delta}_i \geq 0 \tag{12c}$$

The equations are trivially integrated in the elastic range to furnish the same results as before. The integration is also simple in the range of contained plastic deformation with initial conditions provided by the results of the elastic range. However, the inequality *(12c) combined with (3a) and* (3c) gives

$$-\dot{v} \leq \dot{u} \leq \dot{v} \tag{13}$$

in the range of contained plastic deformation. The remaining governing equations place no further restrictions on u. Thus we see again that the solution is not unique.

Physically, the lack of uniqueness stems directly from the lack of a unique strain for the yield stress in the elastic-perfectly plastic material model. This is expressed by the inequality (12c). If the constitutive equation were an equality, as in a material with strain hardening, this lack of uniqueness would not arise. The inequality is on the rate of extension of the bar, which is a combination of $\dot{u}$ and $\dot{v}$.

Now $\dot{v}$ satisfies the strict inequality for $\dot{u} = 0$ and in effect biases the expression in favor of being satisfied. Values of $\dot{u}$ up to the extent of the bias are acceptable. This is the range of possible values for $\dot{u}$.

The lack of uniqueness occurs in variables which are not of primary interest in the problem. The stresses, the external work, and the yield point load are all uniquely determined This last result appeals to one's intuition and is in fact a consequence of the limit analysis theorems.[6]

Certain small changes in the model may alter it so that unique results are obtained. For example, an infinitesimal amount of strain hardening will lead to the unique result u = 0. However, if the angle of OA is reduced slightly while that of OB remains at 45°, then $u = (H/AE)(P - P_e)$ whereas a slight increase in the angle of OA will lead to $u = -(H/AE)(P - P_e)$. Including the non-linearity resulting from applying the loads to the deformed structure gives a unique solution when the load is tensile and a non-unique solution when the load is compressive.[4] Other small changes, such as changing the length of bar OB or changing the angle of both diagonal bars by the same amount will still lead to non-uniqueness. Therefore, since the different modifications lead to different solutions, the lack of uniqueness of the symmetric perfectly plastic case is a real phenomenon.

Mathematically, an essential feature *of the non-uniqueness* phenomenon is associated with the fact that the equilibrium equation (2a) is homogeneous. In rate form it becomes

$$\dot{F}_1 - \dot{F}_3 = 0 \qquad (14)$$

Thus, when bars 1 and 3 yield simultaneously, it follows from (12c) alone that $\dot{F}_1 = \dot{F}_3 = 0$ and (14) becomes redundant. More generally, for any redundant truss with a single parameter load system, the equilibrium equations can always be written so that at least one of them is homogeneous. If all of the bars appearing in the homogeneous equation except bar k yield, the equation predicts $\dot{F}_k = 0$ regardless of the value of $\dot{\delta}_k$. If bar k is elastic, this implies $\dot{\delta}_k = 0$. However, if bar k yields at the same load as the last other bar in the equation, then $\dot{F}_k$ is already known to be zero and the equilibrium equation is redundant. The simultaneous yielding of the last two bars is the simultaneous occurrence of two collapse mechanisms for a portion of the structure. Either mechanism is sufficient to determine the force which this substructure can support.

If a truss is being analysed with a computer program, the onset of non-uniqueness will cause problems since the stiffness matrix will be singular. The program may interpret this to be a collapse mechanism. If the program uses pivoting to reduce round-off error in the solution of the linear equations, the solver may be stopped by a zero pivot value.

There are at least two techniques which can be used to avoid this difficulty. If the program has been written to include the effects of strain hardening, the use of a small value for the strain hardening coefficient, for instance $E \times 10^{-5}$, will make the solution unique without noticeably affecting the unique part of the results. NonSAP[5] is a finite element

program capable of handling non-linear truss problems. With an elastic-perfectly plastic material model, the program stops after the yielding of bars OA and OC. The error message produced is "stop because zero pivot." If a strain hardening coefficient of $E \times 10^{-5}$ is input, the program runs until the yield of bar OB. At that point the equilibrium iterations do not converge and the error message is "iteration limit reached."

For programs written to analyze trusses which are based on the elastic-perfectly plastic material model, the technique is slightly more involved. The usual scheme of such a program is a step by step approach. A load is applied and the elastic problem solved. On the basis of this solution, the bar or bars which yield first are determined. These bars are replaced by the forces acting in them at yield and another elastic analysis is done. The process continues until the stiffness matrix becomes singular. If the truss has a non-unique solution, the stiffness matrix will become singular before the yield point load is reached.

A possible remedy is to replace only one bar at each step, and to infinitesimally increase the yield stress for other bars which were predicted to yield. For the three bar truss, OA and OC are found to yield at the same load. The program arbitrarily picks one of these bars, say OA, and replaces it with the force in the bar at yield. The yield stress in bar OC is increased slightly. In the new elastic solution, the stress in OC does not increase and the next bar to yield is OB. This gives a singular stiffness matrix and is correctly recognized

as collapse. This modification will also correctly handle two bars yielding concurrently which do not represent a case of non-uniqueness.

Several characteristics of this lack of uniqueness in the three bar truss can be generalized. First, the stresses, the external work, and the yield point load are uniquely determined. Second, each of the displacements which is not uniquely determined is bounded. Third, one portion of the structure must yield in such a fashion as to allow more degrees of freedom than are specified by the surrounding material. Mathematically, this last requirement amounts to two or more members yielding simultaneously to reduce an equilibrium equation to an identity.

IV. FURTHER EXAMPLES

The present section considers several other simple structures where non-uniqueness occurs in the range of contained plastic deformation. Only the salient features of the solutions are given here; the details may be found in the appendix.

The truss in figure 2 is composed of bars having the same area, Young's modulus, and yield stress. The loads P are increased monotonically.

This truss is reasonable in appearance and might actually occur in practice. There is no reason for a practising engineer to suspect that the analysis of this truss will present special problems.

At the elastic limit $P = P_e$, bars 1 and 2 both yield in compression. There is no collapse mechanism associated with this load. Indeed, the only conceivable mechanism is a rotation of the upper triangle about the point A which requires either bar 1 or bar 2 to unload.

As the load is further increased, the vertical displacement of point A is determined by the stresses in bars 6 and 7. The upper diamond is free to rotate about point A within the constraint that the magnitude of the strain in bars 1 and 2 increase monotonically. The yield point load P_y is reached when the diagonal bars yield. At this point there is a collapse mechanism.

Mistaking the load P_e when bars 1 and 2 yield for the yield point load of the structure leads to underestimating the strength of this truss by 24%. For example, if the area of

the bars is 1 in^2, $E = 30{,}000$ ksi, and $\sigma_y = 30$ ksi, then 1 and 2 will yield when P reaches 39 kips, whereas the diagonal bars will yield only when P reaches 51 kips.

This example demonstrates the interesting result that the individual displacements under external loads at B and C are bounded but are not uniquely determined. However, the average displacement $v = (v_B + v_C)/2$ and the total external work $W_{ext} = Pv_B + Pv_C$ are unique, even though the individual deflections are not.

A second example concerns a rectangular frame where each member transmits axial and bending forces. Deformations due to bending moments dominate and deformations due to axial forces are neglected. The effect of axial forces on the yield strength of the members is also neglected.

The frame shown in figure 3 has a non-unique solution in the range of contained plastic deformation. All of the members have the same cross-sectional and material properties: E, I, and yield moment M_o. At the support D, the member is fixed against horizontal, vertical, and angular displacements. At A, the member is fixed against vertical displacements. The load P is monotonically increased from zero.

In rate form, the constitutive equations are:

$$M_i^2 \le M_o^2 \qquad (15a)$$

$$\text{If} \quad M_i^2 \le M_o^2 \quad \text{or} \quad M_i\dot{M}_i < 0 \quad \text{then } \dot{\theta}_i = 0 \qquad (15b)$$

$$\text{Else} \quad \dot{M}_i = 0 \quad \text{and} \quad M_i\dot{\theta}_i \ge 0 \qquad (15c)$$

Static considerations independent of material properties

require that the bending moment in BC be constant along the length of the bar. At $P = P_e$, this moment reaches the yield moment, M_o.

For convenience, we assume that the joints are slightly weaker than the adjacent beams so that yield hinges form only at B and C as P is further increased. The moments at these points are then known to be M_o. Equilibrium equations determine the moment distribution throughout the beams AB and BC so long as (15c) holds. This stress distribution is no longer a function of the load P. The incremental deformations in this portion of the frame are due solely to the plastic hinges. At these hinges, the moment-rotation relationship is non-unique.

In the range of contained plastic deformation, the incremental vertical displacement and rotation of CD at the point C are given by the elastic solution to the cantilever beam CD loaded by $(P - P_e)$. Displacements in the mechanism A-B-C are governed by the constitutive equations (15c) and the principle of virtual work:

$$W_{ext} = M_o(\theta_C - \theta_B) \qquad (16)$$

W_{ext} is known. Equations (15c) and (16) do not have a unique solution for θ_B and θ_C.

Although this analysis is based on the assumption that hinges form only at joints, the occurrence of the phenomenon does not depend on this assumption. One point of view is that in the structure being modeled, the joints actually are weaker than the beams. In this case the assumption that hinges occur

at the joints is necessary to represent the physical situation. A second point of view is to assume that the beam yields everywhere the moment reaches the yield moment. This implies that the entire beam BC is plastic. For a perfectly plastic material, angular displacement is independent of bending moment. There is not enough information to determine whether angualr displacements are concentrated at one or more hinges, or whether the rotations are continuously distributed along the beam. Thus, this premise also leads to a non-unique solution with, in fact, infinite degrees of freedom.

A more conventional frame exhibiting a lack of uniqueness is shown in Figure 4. Plastic hinges form at B, then at C for P_e and P_1, respectively. The solution is unique until hinges form simultaneously at A and D for $P = P_2$. For $P_2 < P < P_y$, the incremental displacements at C will be determined by the elastic beams CE and EF whereas the displacement of A will be bounded but not uniquely determined. Physically, the ambiguity is explained by observing that the vertical deflection of C can be accommodated by either rotations at A and C or by a vertical deflection of A and rotations at B and D.

As a final example we consider the two orthogonal intersecting circular arches shown in Figure 5. Arch CD rests on arch AB with a point of smooth contact at E so that only a vertical force is transmitted there. Both arches are pin-supported and have the same elastic properties, but AB is made of a material with a lower yield stress than that of CD.

As the load P is increased, at some value P_1 three hinges will form in AB while CD is still elastic. For $P > P_1$ the

vertical displacement v_E of the center will be determined solely by the properties of arch CD and the increment of the load P.

Since CD offers no resistance to small deflections of point E in the x direction, arch AB is the only source of information about u_E. A solution of the collapse load problem for AB will give the location of the plastic hinges and the moments and axial forces at those hinges. As the load P is increased beyond P_1, AB responds as a mechanism and the solution for u_E must come from studying this mechanism with a known vertical force and displacement. However, this mechanism leads to a singular equation which does not determine u_E. As in the other examples, the inequality in the constitutive equations does lead to bounds on u_E.

V. CONCLUSIONS

In each of these examples, a portion of the structure reaches its load carrying capacity at a load below the yield point load for the entire structure. If there is only one collapse mechanism for the substructure, all quantities are determined. If two collapse mechanisms are possible, the solution may be non-unique. Simultaneous yielding at two or more points is a characteristic of this type of non-uniqueness. This concurrent yielding occurs naturally in symmetric structures, but is also possible in non-symmetric ones.

Despite the lack of uniqueness, certain parameters are determined. All of the stresses are determined. The total external work, and hence the yield point load, is determined. All of the displacements are not determined, but they are bounded. These bounds increase monotonically, are largest at the yield point load for the entire structure, and are the same order of magnitude as the elastic displacements.

The elastic-perfectly plastic material assumption lies at the root of this lack of uniqueness. Even when the history is given, the stress does not uniquely determine the strain. This allows displacements to be independent of stresses. Kinematic arguments may exclude the phenomenon from certain classes of problems, but these are special cases. This paper has demonstrated non-uniqueness in trusses, frames, and arches. Interest in the topic arose because a proposed slip model for finite-element plasticity[7] encountered a lack of uniqueness in a plane strain problem. This type of non-uniqueness must be

considered in all problems involving infinitesimal strains and an elastic-perfectly plastic material model.

Further investigation is needed to ascertain the impact of this phenomenon on the solution of practical problems. Examples of more complex structures and of more realistic structures exhibiting a lack of uniqueness may lead to a more useful criterion for uniqueness. The scope of the examples should be broadened to include systems of plates in bending, plane stress, and plane strain.

Current computer programs need to be tested on structures known to lack uniqueness to see what problems arise and to learn to identify non-uniqueness when it shows up in other structures. It is particularly important to learn to distinguish between a lack of uniqueness and the yield point load, which may not be a simple matter in the case of plane stress and plane strain problems.

Techniques which circumvent the problems caused by non-uniqueness need to be developed and applied to individual programs. Existing programs should be modified so as to take this phenomenon in stride and reach the correct solution in spite of it.

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VII. APPENDIX

The appendix provides the detailed analyses showing each of the examples in Section IV to have a non-unique solution. For each example the elastic solution and a kinematic analysis of the contained plastic deformation are given.

First we consider the truss shown in figure 2. The consitutive equations are given by equation (12) and are trivially integrated in the elastic range to give

$$\text{If } F_i^2 < Y_i^2 \text{ then } \Delta_i = (L_i/AE)F_i \tag{A1a}$$

For the case where no unloading occurs, it is also easy to integrate in the plastic range. The values at the elastic limit furnish initial conditions, and we obtain

$$\text{If } F_i^2 = Y_i^2 \text{ then } F_i = \pm\sigma_y A_i \text{ and } F_i\Delta_i \geq (L_i/AE)F_i^2 \tag{A1b}$$

where the plus or minus sign must be chosen to match the sign of F_i at the elastic limit, and $i = 1, 2, \ldots 7$.

Compatibility is assured if the extensions of the bars are defined in terms of the vertical and horizontal displacements of A, B, and C:

$$\Delta_1 = v_B \quad \Delta_2 = v_C \quad \Delta_3 = u_C - u_B$$

$$\Delta_4 = (u_A - u_B + v_B - v_A)/\sqrt{2}\ , \quad \Delta_5 = (u_C - u_A + v_C - v_A)/\sqrt{2}$$

$$\Delta_6 = (u_A + v_A)/\sqrt{2} \qquad \Delta_7 = (v_A - u_A)/\sqrt{2} \tag{A2}$$

Equilibrium at joints A, B, and C provides six equations which may be rewritten as

$$F_1 - F_2 = F_4 + \sqrt{2}\,F_3 = F_5 - F_4 = F_6 - F_4 = F_7 - F_4 = 0 \tag{A3a-e}$$

$$F_1 + F_4/\sqrt{2} = -P \tag{A3f}$$

With Δ_i defined by (A2), equations (A3) and the appropriate (A1) provide thirteen equations to determine seven forces and six displacements.

In the elastic range, branch (A1a) holds for all bars and the resulting unique solution is

$$F_1 = F_2 = -P(1+4\sqrt{2})/(4\sqrt{2}+3) \quad F_3 = 2P/(4\sqrt{2}+3)$$

$$F_4 = F_5 = F_6 = F_7 = -2\sqrt{2}P/(4\sqrt{2}+3)$$

$$u_A = 0 \quad u_B = -u_C = -F_3\,H/AE$$

$$v_A = 2HF_7/AE \quad v_B = v_C = 2\,H\,F_2/AE \tag{A4}$$

As P is increased, (A4) will hold until

$$P = P_e \equiv A\,\sigma_y(4\sqrt{2}+3)/(1+4\sqrt{2}) \tag{A5}$$

at which point both bars 1 and 2 yield. However, as pointed out in Section IV, no collapse mechanism can be associated with this load.

For larger values of P, branch (A1b) holds for bars 1 and 2 which implies that (A3a) is satisfied identically. The remaining (A3) provide a unique solution for the forces:

$$F_1 = F_2 = -\sigma_y A \quad F_3 = P - \sigma_y A$$

$$F_4 = F_5 = F_6 = F_7 = -\sqrt{2}(P - \sigma_y A) \tag{A6}$$

Bars 3 through 7 are still elastic, but equations (A1a) provide only 5 five equations for the six displacements. The results can be written

$$u_A = 0 \quad v_A = -2\sqrt{2}(H/E)(P/A - \sigma_y) \tag{A7a,b}$$

$$u_B = (1/2)(v_B - v_C - v_A/\sqrt{2}) \tag{A7c}$$

$$u_C = (1/2)(v_B - v_C + v_A/\sqrt{2}) \tag{A7d}$$

$$v_B + v_C = (4 + 1/\sqrt{2})v_A \tag{A7e}$$

Equations (A7) must be supplemented by the inequalities in (A1b) for bars 1 and 2:

$$v_B \leq -2A\,\sigma_y/E \quad v_C \leq -2A\,\sigma_y/E \tag{A7f,g}$$

It is easily seen that (A7e,f,g) have a solution for any

$P \geq P_e$ but that the solution is not unique for $P > P_e$.

This solution remains valid until the four diagonal bars yield at

$$P = P_y \equiv A\ \sigma_y (1+1/\sqrt{2}) \tag{A8}$$

At this load the truss collapses with multiple degrees of freedom.

The elastic solution to the frame in figure 3 can be computed using any of several well known techniques. Since the system is only one degree indeterminant, Castigliano's Theorem is a convenient tool. We first express all moments in terms of the vertical reaction V at A, the load P, and the horizontal distance x from A:

$$\begin{aligned} &AB: \quad M = Vx \\ &BC: \quad M = VH \\ &CD: \quad M = Vx - P(x-H) \end{aligned} \tag{A9}$$

Then, since the vertical displacement vanishes at A,

$$v_A = \frac{\partial U^*}{\partial V} = \int \frac{M}{EI} \frac{\partial M}{\partial V}\, ds = 0 \tag{A10}$$

When this integral is computed using (A9) for M, we obtain

$$V = 11P/16 \tag{A11}$$

In order to see more clearly how this example fits into the general case of non-uniqueness, it is helpful to take a slightly different point of view. Let the moments at the joints be the fundamental unknowns. The frame is loaded with concentrated forces, therefore the moments in the beams vary linearly and can easily be constructed if the moments at the joints are known. Further, linearity implies that the largest moment occurs at a joint. Moment equilibrium of the three beams provides the three equations

$$M_B = VH \tag{A12a}$$

$$M_B - M_C = 0 \tag{A12b}$$

$$M_D - M_C = (P-V)4H \tag{A12c}$$

The solution to this system of three equations, together with (A11), gives the moments as a function of P.

As P is increased, (A11) holds until

$$P = P_e \equiv 16\ M_o/11H \tag{A13}$$

at which point $M_B = M_C = M_o$. The moment along BC is constant and equal to M_o. For convenience we assume, as discussed in Section IV, that yield hinges form only at the joints.

For $P > P_e$, branch (15c) of the constitutive equation holds for the joints B and C. Therefore $\dot{V} = 0$, (A12b) is identically satisfied, and

$$\dot{\theta}_B \geq 0 \qquad \dot{\theta}_C \geq 0 \tag{A14}$$

where θ_B is positive if the angle at B closes and θ_C is positive if the angle at C opens. Once again we observe the key to a non-unique solution: simultaneous yielding at two points causes one of the equilibrium equations to be satisfied identically.

Since V is a constant, the moments in AB and BC are also constant and the additional deformations in these beams are confined to the plastic hinges. These two beams form a mechanism in the sense that the deformations are independent of the load. However, this is not a collapse mechanism because the displacements are limited by the elastic beam CD. Specifically, the vertical deflection δ and the rotation β of CD at the point C are given by the elastic solution to the cantilever beam CD loaded by $(P-P_e)$.

In figure A1 the dotted lines symbolically represent this mechanism at $P = P_e$ and the solid lines represent the mechanism at some P between P_e and P_y. If only the first powers of displacements are considered, changes in the angles at points B and C are

$$\theta_B = -z/H + \delta/H$$

$$\theta_C = z/H + \beta \tag{A15}$$

Substitution of the elastic cantilever solution for CD in in (A15) and the result in (A14) shows that the horizontal

velocity of point A is subject only to

$$-8\dot{P} \leq EI\,\dot{z}/H^2 \leq (64/3)\dot{P} \tag{A16}$$

which clearly has a non-unique solution.

The frame in figure 4 is a symmetric structure, symmetrically loaded. This requires symmetry of stresses, but not symmetry of displacements. Specifically, moments vanish along AB, points with the same letter have the same moment, moments are linear between the lettered points, and the stresses throughout are defined by the seven moments M_A through M_G. The structure is five degrees indeterminant.

The two independent equilibrium equations can be written

$$M_A + M_B - M_C - M_D = 0$$

$$M_B + M_C - M_D - M_E = P\ell \tag{A17}$$

where moments are positive if they produce compression on the top or outside.

The solution for the moments can follow the procedure in the preceding example. The processes is straight forward but tedius. The first plastic hinge forms when $M_B = M_o$ at

$$P = P_E \equiv 3.158\ M_o/\ell \tag{A18}$$

A unique solution continues to exist while P is further increased until the second hinge forms when $M_C = M_o$ at

$$P = P_1 \equiv 3.375\ M_o/\ell \tag{A19}$$

Equations (A17) in rate form are now

$$\dot{M}_A - \dot{M}_D = 0 \tag{A20a}$$

$$\dot{M}_D + \dot{M}_E = -\dot{P}\ell \tag{A20b}$$

The frame is still three degrees indeterminant. As the load is further increased, the next hinges form when $M_A = -M_D = M_o$ at

$$P = P_2 \equiv 3.900\ M_o/\ell \tag{A21}$$

From this point onward, (A20a) is identically satisfied and M_E is given by (A20b) even though the frame is still one degree indeterminant. The next hinge forms at E, and with it a collapse mechanism, at

$$P = P_y \equiv 4.000\ M_o/\ell \tag{A22}$$

Between P_2 and P_y, beams AC, AB, and BD respond as a mechanism. The kinematics suggested by figure A2 and the constitutive equation (15c) lead to

$$\ell\,\dot{\theta}_A = \dot{\delta}_A - \dot{\delta}_C \leq 0 \qquad \ell\,\dot{\theta}_B = \dot{\delta}_A \geq 0$$

$$\ell\,\dot{\theta}_C = \ell\dot{\beta} - \dot{\delta}_A + \dot{\delta}_C \geq 0 \qquad \ell\,\dot{\theta}_D = -\dot{\delta}_A \leq 0 \tag{A23}$$

The yield hinges at B and D imply that no additional load

can be carried by the inner frame. Therefore, the increment of load above P_2 is carried entirely by the elastic structure CEF. The solution of this structure gives:

$$\dot{\delta}_C = 5\ \dot{P}\ell^3/6EI \qquad \dot{\beta} = \dot{P}\ell^2/EI \tag{A24}$$

Therefore the four inequalities (A23) are satisfied provided only that

$$5\dot{P}\ell^3/6EI \geq \dot{\delta}_A \geq 0 \tag{A25}$$

Thus we see that the solution exists but is not unique.

For the orthogonal arches in figure 5, non-uniqueness arises when arch AB has yielded, but the stronger arch CD has not. For small loads P, each of the two arches is fully elastic and carries one half of the total load. The solution to the collapse of a circular arch[6] predicts that the first hinge forms in the center. From this load onward, arch AB will support a load of somewhat less than half of P, say P'. As P is further increased, P' increases to a maximum value of P_y', the yield-point load of arch AB considered as an isolated structure. At this point the arch AB can support no more load, the stresses in AB do not change as P is increased, and further increments in P are carried entirely by arch CD.

When P' reaches P_y', two yield hinges form simultaneously at E and G. Up to this load the solution is unique, and therefore symmetric. Since the stresses in AB do not change as P increases further, they are symmetric and unique for all

values of P.

The details of the kinematic solution depend upon the particular yield relation between moment and axial force, but general features such as uniqueness are independent of this relation. For simplicity of exposition we consider here the piecewise-linear relation

$$|M|/M_o + |N|/N_o \leq 1 \tag{A26}$$

of an idealized I-beam or sandwich section. The plastic flow law then requires

$$\Delta_{1L}/L = -h\,\alpha_L \qquad \Delta_{1R}/L = -h\,\alpha_R$$

$$\Delta_{2L}/L = h\,\beta_L \qquad \Delta_{2R}/L = h\,\beta_r$$

$$h = M_o/L\,N_o \tag{A27}$$

where the kinematic variables are defined in figure A3 and h is a dimensionless constant.

The constitutive relations require that the rate of work done at each hinge be positive. This leads directly to

$$\dot{\alpha}_L + \dot{\alpha}_R \geq 0\ , \quad \dot{\beta}_L \geq 0\ , \quad \dot{\beta}_R \geq 0 \tag{A28}$$

The vertical and horizontal deflection of the point E can be computed starting from either of the pin supports A or B:

$$v/L = (\beta_L - \alpha_L)\sin\phi_0 - (1+h)\beta_L \sin(\phi_0 - \phi)$$

$$= (\beta_R - \alpha_R)\sin\phi_0 - (1+h)\beta_L \sin(\phi_0 - \phi)$$

$$u/L = -(\beta_L - \alpha_L)(1-\cos\phi_0) + \beta_L[1-(1+h)\cos(\phi_0-\phi)] + h\,\alpha_L$$

$$= (\beta_R - \alpha_R)(1-\cos\phi_0) - \beta_R[1-(1+h)\cos(\phi_0-\phi)] - h\,\alpha_R \tag{A29}$$

These four equations can be solved for $\dot{\alpha}_L$, $\dot{\alpha}_R$, $\dot{\beta}_L$, and $\dot{\beta}_R$ in terms of the velocities. Substitution of the results into (A28) shows that the only requirement on the horizontal velocity is

$$\dot{v} \leq \frac{\dot{u}\sin\phi_0}{1+h-\cos\phi_0} \leq -\dot{v} \tag{A30}$$

provided that

$$K \equiv [(1-h)\cos(\phi_0 - \phi) - \cos\phi_0]\sin\phi_0 -$$
$$(1+h-\cos\phi_0)[\sin\phi_0 - (1+h)\sin(\phi_0-\phi)] \geq 0 \tag{A31}$$

It can be shown that (A31) holds for all reasonably small h and for all ϕ_0 less than 90°.

Since $\dot{v}$ (which can be expressed in terms of $\dot{P}$ from the solution for arch CD) is necessarily negative, we see that a non-unique solution exists for this problem also.

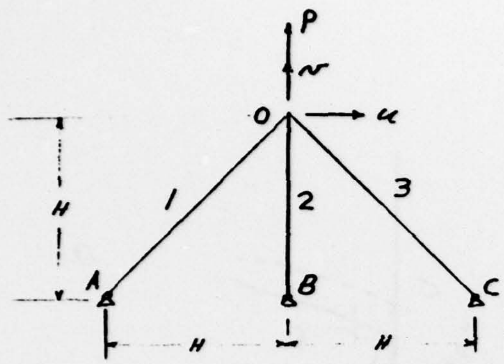

FIG 1

Three-bar truss

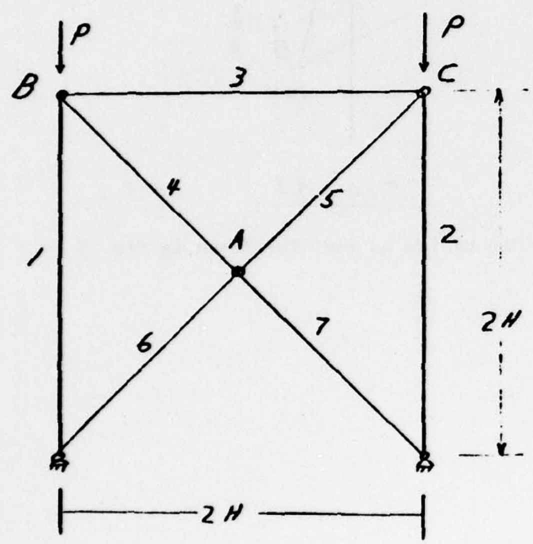

FIG 2

Conventional truss

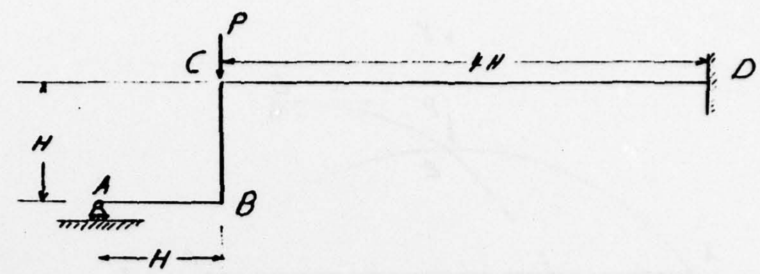

FIG. 3

Simple frame

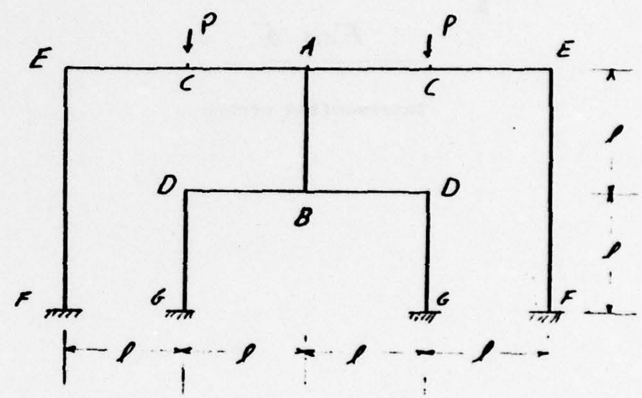

FIG. 4

Conventional frame

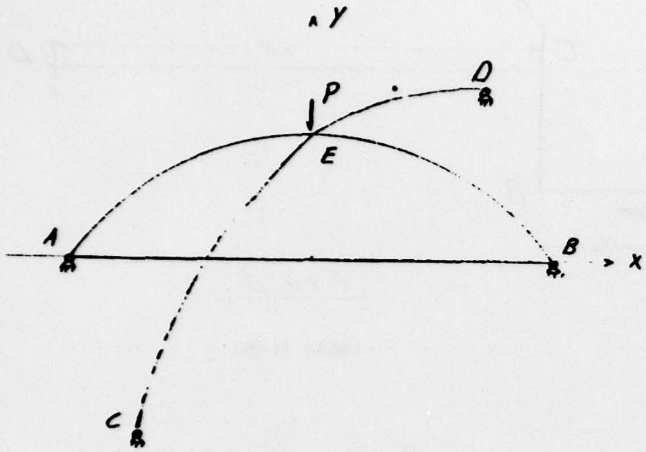

FIG. 5

Intersecting arches

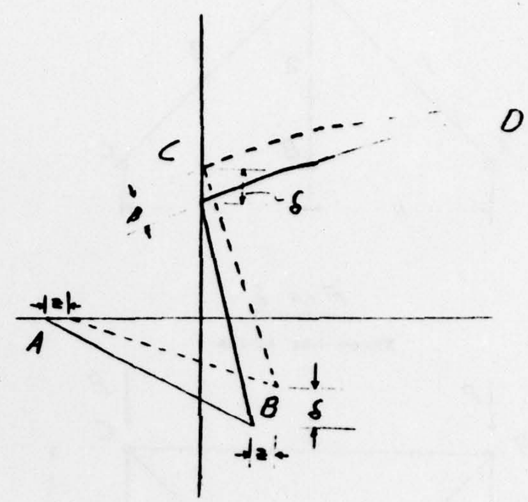

FIG A1

"Mechanism motion" for frame in Fig. 3

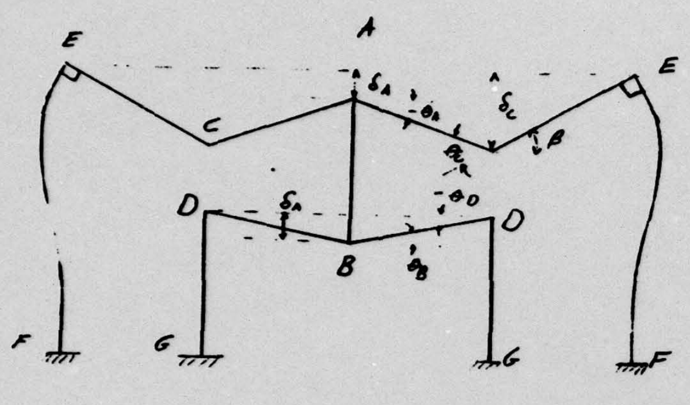

FIG A2

"Mechanism motion" for frame in Fig. 4

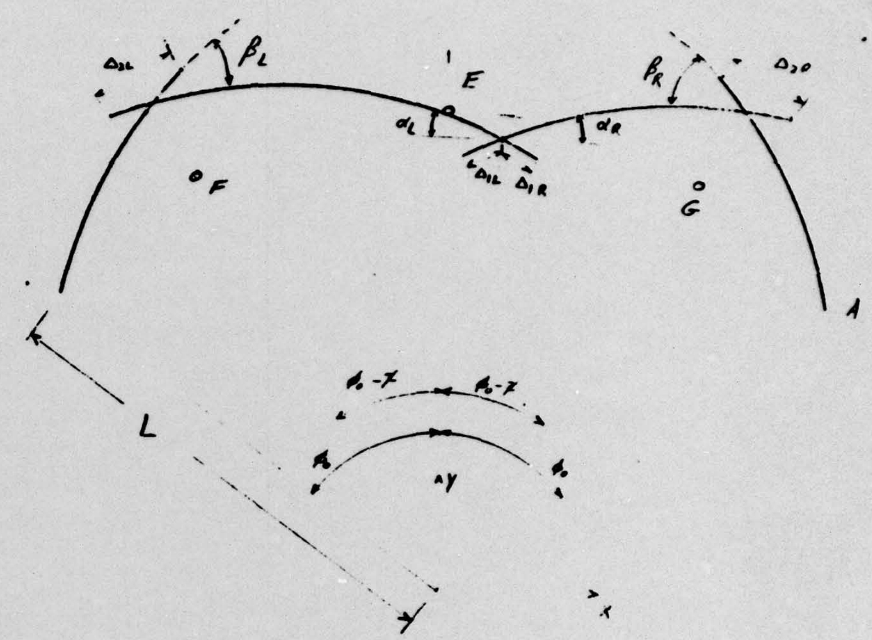

FIG A3

"Mechanism motion" for arch AB in Fig. 5